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AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

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[Continued from the August-September Number.]

CHAPTER VIII.

INNER PRODUCTS,—NORMAL SYSTEMS,—PROJECTION.

117. DEFINITION.—The Inner Product of two *units* of any order is the relative product of the first and complement of the second.

Thus the inner product of E and F is $[E|F]$.

Note.—Grassmann seems to have regarded the outer (52) and inner products as different in nature. But they both obey the laws of combinatory multiplication, the complement sign indicating a preliminary change to be made in the factor following it before it is combined with the other.

118. *The inner product of any two quantities is equal to the relative product of the first and complement of the second.*

PROOF.—Let $A=\alpha A_1+...+\alpha_n A_n$, $B=\beta B_1+...+\beta_n B_n$, where $A_1, ...,$ $B_1, ...,$ are units. Also for the moment let $\times$ signify the inner product.

$$\text{Then } [A\times B]\equiv[(\alpha_1 A_1+...+\alpha_n B_n)\times(\beta_1 B_1+...\beta_n B_n)]$$

$$=\Sigma\alpha_r\beta_s[A_r\times B_s].\ (28)$$

Now since $A_1, \ldots, B_1, \ldots$, are *units*, $[A_r \times B_s]=[A_r \mid B_s]$. (117)

Then $[A \times B]=\Sigma\alpha_r\beta_s[A_r \mid B_s]=\Sigma[\alpha_r A_r . \Sigma\beta_s \mid B_s]$ (28)

$$\equiv[A\Sigma\beta_s \mid B_s]=[A \mid \Sigma\beta_s B_s] \text{ (58) } \equiv[A \mid B].$$

119. *The inner product of two quantities of the same order is a number.* For, letting r denote the order of each factor, the complement of the second factor is of order n-r, and the product of the first factor which is of the order r and another which is of order n-r is of the nth order, *i. e.* is a pure number. (61)

Corollary.—*On account of the scalar value of the product, in this case* $[A \mid B]=[B \mid A]$.

120. *The inner product of two equal units is unity, while that of two different units of the same order is zero.*

Thus $[E_1 \mid E_1]=1$ (57), $[E_r \mid E_s]=0$. (43)

121. *If* $E_1, \ldots, E_n$ *are units of any order, but all of the same order, then*

$$[A \mid B]\equiv[(\alpha_1 E_1+\ldots+\alpha_n E_n) \mid (\beta_1 E_1+\ldots+\beta_n E_n)]$$

$$=\alpha_1\beta_1+\alpha_2\beta_2+\ldots+\alpha_n\beta_n. \text{ (120)}$$

122. *If* $B=A$ *in* 121, *we get what is called* the inner square of A, which is denoted by A^{2}; thus we have

$$A^2=\alpha_1^2+\alpha_2^2+\ldots\alpha_n^2.$$

123. Normal Systems.—Definition.—The numerical value of an extensive quantity A is defined as the positive square root of the inner square of A. This definition reminds one of the modulus in complex numbers.

124. Definition.—Two quantities (which do not equal zero) are said to be normal to each other if their inner product is zero. Two spaces are said to be every way (allseitig) normal to each other when each quantity in either space is normal to every quantity in the other space.

125. Definition.—A normal system of the nth order is a set of n numerically equal quantities of the first order of which each is normal to every other. If at the same time n is the order of the space, then such quantities constitute a *perfect* normal system. The numerical value of these n quantities is at the same time the numerical value of the system. Every normal system whose numerical value is unity is called a *simple* system.

126. Definition.—By Circular Alteration is meant that transformation of a system by which two quantities a and b of the system are transformed respectively into $xa+yb$ and $\pm(xb-ya)$, where $x^2+y^2=1$. The circular alteration is said to be positive or negative according as $+$ or $-$ is taken in the double sign.

127. *By circular alteration any normal system is transformed into another normal system having the same numerical value.*

PROOF.—Suppose $a, b, \ldots$ to be the quantities of a normal system. Then, by definition,

$$0=[a \mid b]=[a \mid c]=[b \mid c]=\ldots, \text{ and } a^2=b^2=c^2=\ldots$$

Let now a change into $a_1=xa+yb$ and b into $b_1=\pm(xb-ya)$ where $x^2+y^2=1$. We are to show that $a_1, b_1, c \ldots$ constitute a normal system. We have

$$a_1{}^2=(xa+yb)^2=x^2a^2+y^2b^2, \text{ since } [a \mid b]=0,$$

$$=(x^2+y^2)a^2=a^2, \text{ by hypothesis.}$$

Similarly, we can prove $b_1{}^2=b^2$.

Also, $[a_1 \mid b_1]=\pm[(xa+yb) \mid (xb-ya)]=\pm xy(b^2-a^2)=0.$

Finally, $[a_1 \mid c]=[(xa+yb)c]=x[a \mid c]+y[b \mid c]=0.$

Hence, by definition, $a_1, b_1, c, \ldots$ constitute a normal system.

128. *The combinatory product of quantities of a normal system is unaltered by positive circular alteration, and has its sign changed by negative circular alteration.*

Using the notation of 127, we have

$$[a_1b_1]=[(xa+yb)(xb-ya)]=x^2[ab]-y^2[ba] \ (34)=(x^2+y^2)[ab]=[ab].$$

129. *All the quantities of a normal system are independent.*

PROOF.—Suppose $a, b, c, \ldots$ to be quantities of a normal system. Let us assume for the moment that they are not independent and that

$$a=\beta b+\gamma c+\ldots$$

We multiply both sides by $|a$. Then

$$a^2=\beta[b \mid a]+\gamma[c \mid a]+\ldots=0. \ (124)$$

But $a^2=0$ contradicts the hypothesis in 124. Hence the quantities of a normal system are independent.

130. *The system of the original units* (11) *is a perfect normal system whose numerical value is unity* (125).

PROOF.—Let $e_1, \ldots e_n$ be the original units. Then (120)

$$e_1{}^2=e_2{}^2=\ldots=e_n{}^2=1, \text{ and } 0=[e_1 \mid e_2]=\ldots$$

131. PROJECTION.—DEFINITION.—If n is the order of the space considered, $a_1, \ldots a_n$ are independent quantities of the first order, $A_1, A_2, \ldots A_n$ are the

multiplicative combinations of these quantities of any one class, $A_1, \ldots A_m$ the multiplicative combinations of m of the same quantities $a_1, \ldots a_m$, and

$$A = \alpha_1 A_1 + \alpha_2 A_2 + \ldots \alpha_m A_m + \ldots \alpha_n A_n$$

$$A' = \alpha_1 A_1 + \alpha_2 A_2 + \ldots \alpha_m A_m.$$

A' is called the projection of A on the space $[a_1, a_2, \ldots a_m]$ by exclusion of the space $[a_{m+1}, \ldots a_n]$.

REMARK.—We have introduced here for want of a better the geometrical term *projection* to translate *Zurückleitung*. Literally *Zurückleitung* means "leading back."

132. The projection A' of a quantity A on a space B by exclusion of the space C is

$$A' = \frac{[B.AC]}{[BC]}.$$

PROOF.—Let the quantities be taken as in 131 and let

$$[a_1 \ldots a_m] = B, \; [a_{m+1} \ldots a_n] = C.$$

Then $[AC] = [(\alpha_1 A_1 + \ldots + \alpha_m A_m + \alpha_{m+1} A_{m+1} + \ldots \alpha_n A_n) C]$.

But since $A_1, \ldots A_m$ are the combinations formed out of $a_1, \ldots a_m$ and $A_{m+1}, \ldots A_n$ those out of $a_1, \ldots a_n$ which are not at the same time combinations out of $a_1, \ldots a_m$, then must each of the quantities $A_{m+1}, \ldots A_n$ contain at least one of the factors of $a_{m+1}, \ldots a_n$, and thus must have a factor in common with C. Therefore the terms

$$\alpha_{m+1} A_{m+1} C, \ldots \alpha_n A_n C$$

are each equal to zero. (43) Hence

$$[AC] = [(\alpha_1 A_1 + \ldots + \alpha_m A_m) C] = \alpha_1 [A_1 C] + \ldots + \alpha_m [A_m C].$$

$$\therefore [B.AC] = \alpha_1 [B.A_1 C] + \ldots + \alpha_m [B.A_m C].$$

Since now each of the quantities $A_1, \ldots A_m$ consists of factors which are contained in B, then is each of the same incident to B. Consequently, since the orders of B and C are together equal to n, by (72), we have

$$[B.A_1 C] = [BC] A_1, \ldots [B.A_n C] = [BC] A_n,$$

and therefore

$$[B.AC] = [BC](\alpha_1 A_1 + \ldots + \alpha_m A_m) = [BC] A'.$$

Now since $[BC]$ is a number, we get

$$A'=\frac{[B.AC]}{[BC]}.$$

133. *If the projections taken in the same sense of the terms of an equation replace those terms, the result is a true equation.*

PROOF.—Let Q be the space on which the projection is made, R that excluded and $[QR]=1$. Then if the given equation is

$$P=\alpha A+\beta B+\ldots$$

$$[PR]=\alpha[AR]+\beta[BR]+\ldots$$

$$\text{and } [Q.PR]=\alpha[Q.AR]+\beta[Q.BR]+\ldots$$

$$\text{or, } P'=\alpha A'+\beta B'+\ldots$$

where P', A', ... are the *projections* of terms in the given equation.

134. DEFINITION.—The projection A' of a quantity A on a space B by exclusion of the space $|B$ is called the *normal* projection.

From 132 we have for the normal projection

$$A'=\frac{[B.(A\,|\,B)]}{B^2}.$$

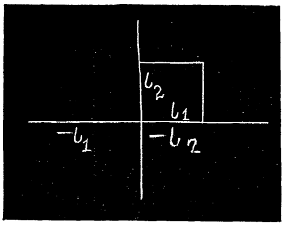
CHAPTER IX.

INNER PRODUCTS, NORMAL SYSTEMS, AND PROJECTION IN GEOMETRY.

135. Let ι_1 and ι_2 be two unit vectors constituting a simple normal system of the second order. Then by definition 125

$$\iota_1\iota_2=1, \text{ and } \iota_1\,|\,\iota_2=0,\ \iota_2\,|\,\iota_1=0.$$

We have also by definition of complement (57) $|\iota_1=\iota_2$ and $|\iota_2=-\iota_1$ since these values make $\iota_1\,|\,\iota_1=\iota_1\iota_2=1$, and $\iota_2\,|\,\iota_2=-\iota_2\iota_1=1$ (37). Also $||\iota_1=-\iota_1$ and $||\iota_2=-\iota_2$ (60).



Thus we see that taking the complement of a vector twice reverses it, *i. e.* revolves it through 180°, so that we are led to suppose that taking it once would revolve the vector through 90°. If this view of the complement can be shown to be consistent with the laws of the *Ausdehnungslehre*, we will adopt it.

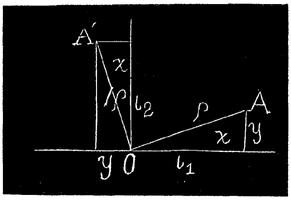
We have introduced above the following equations whose geometrical interpretation we append to each.

(1) $\iota_1\iota_2=1=$the unit of area (88).

(2) $|\iota_2=-\iota_1$, *i. e.* taking the complement of ι_2 revolves it in the positive direction, opposite to the motion of the hands of a watch, into $-\iota_1$.

(3) $\iota_1 | \iota_2=\iota_1(-\iota_1)=0$ (34).

(4) Let $\rho=x\iota_1+y\iota_2$ be any vector in the plane. Then, 58,



$$|\rho=x|\iota_1+y|\iota_2=x\iota_2-y\iota_1.$$

The last value shows that $|\rho$ is OA', at right angles to OA. Thus here again taking the complement of a vector revolves it through 90° in the positive direction.

136. Comparing now the last part of the preceding article with 126-127 we see that the system whose units are ι_1 and ι_2 is transformed by circular alteration into that whose units are ρ and $|\rho$, provided $x^2+y^2=1$, which makes the tensors of the new vectors each equal to unity. Thus circular alteration turns each of the units through the same angle in the same direction.

137. *If* ε_1 *and* ε_2 *are any two vectors,* $\varepsilon_1 | \varepsilon_2=0$ *is the condition that these two vectors are perpendicular to each other.*

For, $|\varepsilon_2$ denotes a vector perpendicular to ε_2 and $\varepsilon_1 | \varepsilon_2=0$ denotes that ε_1 and $|\varepsilon_2$ coincide.

138. Let ι_1, ι_2, ι_3 be three unit vectors constituting a simple normal system of the third order. Then by Definition 125

$$\iota_1\iota_2\iota_3=1,\quad \iota_1 | \iota_2=0,\quad \iota_1 | \iota_3=0,\quad \iota_2 | \iota_3=0.$$

We have also, by definition of complement (57),

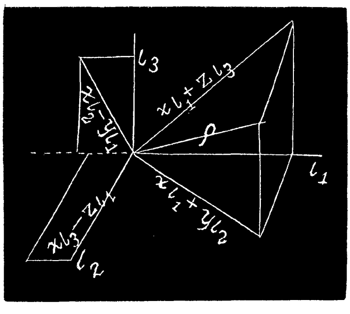
$$|\iota_1=\iota_2\iota_3,\ |\iota_2=\iota_3\iota_1,\ |\iota_3=\iota_1\iota_2\ ;\ ||\iota_1=\iota_1,\ ||\iota_2=\iota_2,\ ||\iota_3=\iota_3\ (60).$$

Thus we see (89) that the complement of a line is a plane, and the complement of a plane is a line.

Let $\rho=x\iota_1+y\iota_2+z\iota_3$=any line in space. Then, (58),

$$|\rho=x|\iota_1+y|\iota_2+z|\iota_3=x[\iota_2\iota_3]+y[\iota_3\iota_1]+z[\iota_1\iota_2]$$

$$=\frac{1}{x}[(x\iota_2-y\iota_1)(x\iota_3-z\iota_1)]\ (38,\ 34).$$



By 89 the right member equals the plane segment formed with $x\iota_2-y\iota_1$ and $x\iota_3-z\iota_1$.

Now ρ is perpendicular to each of these vectors and therefore perpendicular to their plane. For

$[(x\iota_1+y\iota_2+z\iota_3) | (x\iota_2-y\iota_1)]=0$, and

$$[(x\iota_1+y\iota_2+z\iota_3)\,|\,(x\iota_3-z\iota_1)]=0,$$

since, by hypothesis, $[\iota_1\,|\,\iota_2]=0$, etc.

Hence the complement of a vector is a plane perpendicular to it.

139. PROJECTIONS.—*Let $\rho=x\varepsilon_1+y\varepsilon_2$ be given to find its projection on ε_1 and ε_2, respectively.*

Expressing ρ as the sum of its projections on ε_1 and ε_2, we have

$$\rho=\frac{[\varepsilon_1.\rho\varepsilon_2]}{[\varepsilon_1\varepsilon_2]}+\frac{[\varepsilon_2.\rho\varepsilon_1]}{[\varepsilon_2\varepsilon_1]}\ (132)=\frac{[\rho\varepsilon_2]}{[\varepsilon_1\varepsilon_2]}\varepsilon_1+\frac{[\rho\varepsilon_1]}{[\varepsilon_2\varepsilon_1]}\varepsilon_2$$

since $[\rho\varepsilon_2]$, and $[\rho\varepsilon_1]$ are scalars in plane space.

140. *To express ρ as the sum of its projections on any three vectors ε_1, ε_2, ε_3.*

$$\rho=\frac{[\rho\varepsilon_2\varepsilon_3]}{[\varepsilon_1\varepsilon_2\varepsilon_3]}\varepsilon_1+\frac{[\rho\varepsilon_3\varepsilon_1]}{[\varepsilon_1\varepsilon_2\varepsilon_3]}\varepsilon_2+\frac{[\rho\varepsilon_1\varepsilon_2]}{[\varepsilon_1\varepsilon_2\varepsilon_3]}\varepsilon_3\ \text{(By 132. See 8).}$$

141. *To express p as the sum of its projections on any four points p_1, p_2, p_3, p_4.*

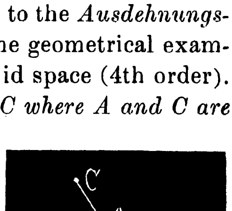
$$p=\frac{[p\ p_2p_3p_4]}{[p_1p_2p_3p_4]}p_1-\frac{[p\ p_3p_4p_1]}{[p_1p_2p_3p_4]}p_2+\frac{[p\ p_4p_1p_2]}{[p_1p_2p_3p_4]}p_3-\frac{[p\ p_1p_2p_3]}{[p_1p_2p_3p_4]}p_4\ (132).$$

See Articles 95 and 9. The substitution of $p\rho$ for p (76) in the last equation may serve to throw light on this case of point projection.

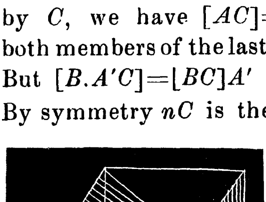
142. Since the formula of 132 is general in its application, the quantities in the equation of 140 may be all points, or all vectors, or all lines, or all plane vectors. In the equation of 141 the points may all be replaced by planes.

143. Following Hermann Grassmann Jr. in his notes to the *Ausdehnungslehre* of 1862, we will illustrate the formula of 132 by some geometrical examples. We suppose the quantities considered situated in solid space (4th order).

(1) *To find the projection A' of A on B by exclusion of C where A and C are points and B is the plane segment, MN.*



A' is the point where CA pierces B taken such that $A=nC+A'$. For, multiplying both members of $A=nC+A'$ by C, we have $[AC]=[A'C]$. Again multiplying B by both members of the last equation, we get $[B.AC]=[B.A'C]$. But $[B.A'C]=[BC]A'$ (72); whence $A'=[B.AC]\div[BC]$. By symmetry nC is the projection of A on C by exclusion of B.



(2) *To find the projection A' of the plane segment A by exclusion of the point C.*

We have $A=A'+P$ where P is in the plane passing through C and the intersection of A and B.

Proof follows lines of (1). Begin by multiplying through by C.

We have also P is the projection of A on C by exclusion of B as in (1).

144. Suppose $[q_2q_3q_4]$ denotes a plane and $[p_1p_2]$ a line. Then their stereometric product is a scaler times their point of intersection (111). Now let

$$[p_1p_2 \cdot q_2q_3q_4] = xp_1 + yp_2 = -[p_2q_2q_3q_4]p_1 + [p_1q_2q_3q_4]p_2$$

by multiplying the members of the first equation by p_2 and p_1 in turn, thus getting values for x and y. Now multiply the members of the last equation by $|q_2$ and at the same time write $[q_2q_3q_4]$ as $|q_1$ (138). Then

$$[p_1p_2\,|q_1\,|q_2] = -[p_2\,|q_1][p_1\,|q_2] + [p_1\,|q_1][p_2\,|q_2]$$

$$\text{or } [p_1p_2\,|q_1q_2] = \begin{vmatrix} p_1\,|\,q_1 & p_1\,|\,q_2 \\ p_2\,|\,q_2 & p_2\,|\,q_2 \end{vmatrix} \quad (55,\ 64).$$

Putting $q_1 = p_1$ and $q_2 = p_2$ we have

$$[p_1p_2]^2 = p_1{}^2p_2{}^2 - [p_1\,|\,p_2]^2.$$

$p_1 = 1$ in the same equation, we have

$$[p_2\,|\,q_1q_2] = [p_2\,|\,q_2] \cdot |\,q_1 - [p_2\,|\,q_1]\,|\,q_2.$$

This equation holds also when the p's are replaced by vectors.

145. Suppose $[p_1p_2p_3]$ denotes a plane and L a line. Then (111)

$$[p_1p_2p_3L] = xp_1 + yp_2 + zp_3 = [p_2p_3L]p_1 + [p_3p_1L]p_2 + [p_1p_2L]p_3$$

by multiplying through by $[p_2p_3]$, $[p_3p_1]$, $[p_1p_2]$ in turn, thus getting values for x, y, z. Now for L put $|\,q_1q_2$ and multiply the members by $|\,q_3$. Then

$$[p_1p_2p_3 \cdot |\,q_1q_2 \cdot |\,q_3] = [p_2p_3\,|\,q_1q_2][p_1\,|\,q_3]$$

$$+ [p_3p_1\,|\,q_1q_2][p_2\,|\,q_3] + [p_1p_2\,|\,q_1q_2][p_3\,|\,q_3],$$

$$\text{or, } [p_1p_2p_3\,|\,q_1q_2q_3] = \begin{vmatrix} p_1\,|\,q_1 & p_1\,|\,q_2 & p_1\,|\,q_3 \\ p_2\,|\,q_1 & p_2\,|\,q_2 & p_2\,|\,q_3 \\ p_3\,|\,q_1 & p_3\,|\,q_2 & p_3\,|\,q_3 \end{vmatrix} \quad (55,\ 64,\ 144).$$

In this equation planes may be substituted for points.

[To be Continued.]